

Biharmonic curves on LP -Sasakian manifolds

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Abstract. In this paper we give necessary and sufficient conditions for spacelike and timelike curves in a conformally flat, quasi conformally flat and conformally symmetric 4-dimensional LP -Sasakian manifold to be proper biharmonic. Also, we investigate proper biharmonic curves in the Lorentzian sphere S_1^4 .

Keywords. Harmonic Maps, Biharmonic Maps, Lorentzian para-Sasakian Manifolds.

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1 Introduction

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by E. Almansi, T. Levi-Civita and M. Nicolescu. Recently, biharmonic functions on Riemannian manifolds were studied by R. Caddeo and L. Vanchke [5, 6], L. Sario, M. Nakai and C. Wang [30].

In the last decade there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations (see [12, 22, 33, 36, 37]), because biharmonic maps are solutions of a fourth order strongly elliptic semi-linear PDE.

Let $C^\infty(M, N)$ denote the space of smooth maps $\Psi : (M, g) \rightarrow (N, h)$ between two Riemannian manifolds. A map $\Psi \in C^\infty(M, N)$ is called *harmonic* if it is a critical point of the *energy* functional

$$E : C^\infty(M, N) \rightarrow \mathbb{R}, E(\Psi) = \frac{1}{2} \int_M |d\Psi|^2 v_g$$

and is characterized by the vanishing of the tension field $\tau(\Psi) = \text{trace} \nabla d\Psi$ where ∇ is a connection induced from the Levi-Civita connection ∇^M of M and the pull-back connection ∇^Ψ . As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by J. Eells and J. H. Sampson in [15]. *Biharmonic maps* between Riemannian manifolds $\Psi : (M, g) \rightarrow (N, h)$ are the critical points of the *bienergy functional*

$$E_2(\Psi) = \frac{1}{2} \int_M |\tau(\Psi)|^2 v_g.$$

The first variation formula for the bienergy which is derived in [20, 21] shows that the Euler-Lagrange equation for the bienergy is

$$\tau_2(\Psi) = -J(\tau(\Psi)) = -\Delta\tau(\Psi) - \text{trace}R^N(d\Psi, \tau(\Psi))d\Psi = 0,$$

where $\Delta = -\text{trace}(\nabla^\Psi \nabla^\Psi - \nabla_{\nabla^\Psi}^\Psi)$ is the rough Laplacian on the sections of $\Psi^{-1}TN$ and $R^N(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ is the curvature operator on N . From the expression of the bitension field τ_2 , it is clear that a harmonic map is automatically a biharmonic map. So non-harmonic biharmonic maps which are called proper biharmonic maps are more interesting.

In a different setting, B. Y. Chen [13] defined biharmonic submanifolds $M \subset R^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = 0$, where Δ is the rough Laplacian, and stated the following

- Conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, that is minimal.

If the definition of biharmonic maps is applied to Riemannian immersions into Euclidean space, the notion of Chen's biharmonic submanifold is obtained, so the two definitions agree.

The non-existence theorems for the case of non-positive sectional curvature codomains, as well as the

- Generalized Chen's conjecture: Biharmonic submanifolds of a manifold N with $Riem^N \leq 0$ are minimal,

encouraged the study of proper biharmonic submanifolds, that is submanifolds such that the inclusion map is a biharmonic map, in spheres or another non-negatively curved spaces (see [7, 9, 16, 19, 28, 29]).

Of course, the first and easiest examples can be found by looking at differentiable curves in a Riemannian manifold. Obviously geodesics are biharmonic. Non-geodesic biharmonic curves are called proper biharmonic curves. Chen and Ishikawa [14] showed non-existence of proper biharmonic curves in Euclidean 3-space E^3 . Moreover they classified all proper biharmonic curves in Minkowski 3-space E_1^3 (see also [18]). Caddeo, Montaldo and Piu showed that on a surface with non-positive Gaussian curvature, any biharmonic curve is a geodesic of the surface [8]. So they gave a positive answer to generalized Chen's conjecture. Caddeo et al. in [7] studied biharmonic curves in the unit 3-sphere. More precisely, they showed that proper biharmonic curves in S^3 are circles of geodesic curvature 1 or helices which are geodesics in the Clifford minimal torus. Then the same authors studied the biharmonic submanifolds of unit n-sphere [9].

On the other hand, there are a few results on biharmonic curves in arbitrary Riemannian manifolds. The biharmonic curves in the Heisenberg group H_3 are investigated in [10] by Caddeo et al. In [16] Fetcu studied biharmonic curves in the generalized Heisenberg group and obtained two families of proper biharmonic curves. Also, the explicit parametric equations for the biharmonic curves on Berger spheres S_ε^3 are obtained by Balmuş in [3].

A generalization of Riemannian manifolds with constant sectional curvature is represented by Sasakian space forms. In particular, a simply connected three-dimensional Sasakian space form of constant holomorphic sectional curvature 1 is isometric to S^3 . So in this context J. Inoguchi classified in [19] the proper biharmonic Legendre curves and Hopf cylinders in a 3-dimensional Sasakian space form and in [17] the explicit parametric equations were obtained. T. Sasahara [31], analyzed the proper biharmonic Legendre surfaces in Sasakian space forms and in the case when the ambient space is the unit 5-dimensional sphere S^5 he obtained their explicit representations.

Other results on biharmonic Legendre curves and biharmonic anti-invariant surfaces in Sasakian space forms and (κ, μ) -manifolds are given in [1, 2].

In this paper we give some necessary and sufficient condition for a spacelike and a timelike curve lying in a 4-dimensional conformally flat, quasi conformally flat and conformally symmetric Lorentzian para-Sasakian manifold to be proper biharmonic.

The study of Lorentzian almost paracontact manifolds was initiated by Matsumoto in 1989 [24]. Also he introduced the notion of Lorentzian para-Sasakian (for short LP -Sasakian) manifold. I. Mihai and R. Rosca [26] defined the same notion independently and thereafter many authors [23, 27, 39] studied LP -Sasakian manifolds.

2 Preliminaries

2.1 Biharmonic maps between Riemannian manifolds

Let (M, g) and (N, h) be Riemannian manifolds and $\Psi : (M, g) \rightarrow (N, h)$ be a smooth map. The tension field of Ψ is given by $\tau(\Psi) = \text{trace} \nabla d\Psi$, and for any compact domain $\Omega \subseteq M$, the bienergy is defined by

$$E_2(\Psi) = \frac{1}{2} \int_{\Omega} |\tau(\Psi)|^2 v_g.$$

Then a smooth map Ψ is called biharmonic map if it is a critical point of the bienergy functional for any compact domain $\Omega \subseteq M$. The first variation formula for the bienergy functional is given by

$$\frac{dE_2(\Psi_t)}{dt} \Big|_{t=0} = \int_{\Omega} \langle \tau_2(\Psi), w \rangle v_g,$$

where v_g is the volume element, w is the variational vector field associated to the variation $\{\Psi_t\}$ of Ψ and

$$\tau_2(\Psi) = -J(\tau_2(\Psi)) = -\Delta^{\Psi} \tau(\Psi) - \text{trace} R^N(d\Psi, \tau(\Psi)) d\Psi.$$

Here Δ^{Ψ} is the rough Laplacian on the sections of the pull-back bundle $\Psi^{-1}TN$ which is defined by

$$\Delta^{\Psi} V = - \sum_{i=1}^m \{ \nabla_{e_i}^{\Psi} \nabla_{e_i}^{\Psi} V - \nabla_{\nabla_{e_i}^M e_i}^{\Psi} V \}, \quad V \in \Gamma(\Psi^{-1}TN),$$

where ∇ is the pull-back connection on the pull-back bundle $\Psi^{-1}TN$ and $\{e_i\}_{i=1}^m$ is an ortonormal frame on M .

From the definition of bienergy and the equation $\tau_2(\Psi)$, some remarks on biharmonic maps are following:

- a map Ψ is biharmonic if and only if its tension field is in the kernel of the Jacobi operator;
- a harmonic map is obviously a biharmonic map;
- a harmonic map is an absolute minimum of the bienergy.

In particular, if the target manifold N is the Euclidean space E^m , then the biharmonic equation of a map $\Psi : M \rightarrow E^m$ is

$$\Delta^2 \Psi = 0,$$

where Δ is the Laplace-Beltrami operator of (M, g) . Also, biharmonic parametrized curves $\gamma : I \subset \mathbb{R} \rightarrow M$ are solutions of the fourth order differential equation

$$\nabla_T^3 T - R(T, \nabla_T T)T = 0.$$

2.2 Lorentzian Almost paracontact manifolds

Let M be an n -dimensional smooth connected paracompact Hausdroff manifold with a Lorentzian metric g , i.e., g is a smooth symmetric tensor field of type $(0, 2)$ such that at every point $p \in M$, the tensor $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \dots, +)$, where $T_p M$ is the tangent space of M at the point p . Then (M, g) is known to be a Lorentzian manifold. A non-zero vector $X_p \in T_p M$ can be spacelike, null or timelike, if it satisfies $g_p(X_p, X_p) \geq 0$, $g_p(X_p, X_p) = 0$ ($X_p \neq 0$) or $g_p(X_p, X_p) < 0$ respectively.

Let M be an n -dimensional differentiable manifold equipped with a triple (ϕ, ξ, η) , where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form on M such that [24]

$$(2.2.1) \quad \eta(\xi) = -1,$$

$$(2.2.2) \quad \phi^2 = I + \eta \otimes \xi,$$

where I denotes the identity map of $T_p M$ and \otimes is the tensor product. The equations (2.2.1) and (2.2.2) imply that

$$\begin{aligned} \eta \circ \phi &= 0, \\ \phi \xi &= 0, \\ \text{rank}(\phi) &= n - 1. \end{aligned}$$

Then M admits a Lorentzian metric g , such that

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

and M is said to admit a Lorentzian almost paracontact structure (ϕ, ξ, η, g) . Then we get

$$\begin{aligned} g(X, \xi) &= \eta(X), \\ \Phi(X, Y) &\equiv g(X, \phi Y) \equiv g(\phi X, Y) \equiv \Phi(Y; X), \\ (\nabla_X \Phi)(Y, Z) &= g(Y, (\nabla_X \phi)Z) = (\nabla_X \Phi)(Z, Y), \end{aligned}$$

where ∇ is the covariant differentiation with respect to g . It is clear that Lorentzian metric g makes ξ a timelike unit vector field, i.e, $g(\xi, \xi) = -1$. The manifold M equipped with a Lorentzian almost paracontact structure (ϕ, ξ, η, g) is called a Lorentzian almost paracontact manifold (for short *LAP*-manifold) [24, 25].

In equations (2.2.1) and (2.2.2) if we replace ξ by $-\xi$, we obtain an almost paracontact structure on M defined by Satō [32].

A Lorentzian almost paracontact manifold M endowed with the structure (ϕ, ξ, η, g) is called a Lorentzian paracontact manifold (for short *LP*-manifold) [24] if

$$\Phi(X, Y) = \frac{1}{2}((\nabla_X \eta)Y + (\nabla_Y \eta)X).$$

A Lorentzian almost paracontact manifold M endowed with the structure (ϕ, ξ, η, g) is called a Lorentzian para Sasakian manifold (for short *LP*-Sasakian) [24] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X,$$

or equivalently,

$$(\nabla_X \phi)Y = \eta(Y)X + g(X, Y)\xi + 2\eta(X)\eta(Y)\xi,$$

or equivalently,

$$(\nabla_X \Phi)(Y, Z) = g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z).$$

In a *LP*-Sasakian manifold the 1-form η is closed.

Also Matsomoto in [24] showed that if an n -dimensional Lorentzian manifold (M, g) admits a timelike unit vector field ξ such that the 1-form η associated to ξ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)Z = g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$

then (M, g) admits an *LP*-Sasakian structure.

An *LP*-Sasakian manifold M^n is said to be η -Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), X, Y \in \Gamma(TM),$$

where a and b are functions on M^n [4, 38].

The conformal curvature tensor C is defined by

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2}\{g(Y, Z)QX - g(X, Z)QY \\ &\quad + S(Y, Z)X - S(X, Z)Y\} + \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned}$$

where $S(X, Y) = g(QX, Y)$. If $C = 0$ then the LP -Sasakian manifold is called conformally flat.

The quasi-conformal curvature tensor \tilde{C} is given by

$$\begin{aligned} \tilde{C}(X, Y)Z &= aR(X, Y)Z + b\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY\} - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned}$$

where a, b constants such that $ab \neq 0$ and $S(Y, Z) = g(QY, Z)$. If $\tilde{C} = 0$ then the LP -Sasakian manifold is called quasi conformally flat. In [34] it was proved that a conformally flat and a quasi conformally flat LP -Sasakian manifold is of constant curvature and the value of this constant is $+1$. Also the same authors showed in [34] that if in an LP -Sasakian manifold M^n ($n > 3$) the relation $R(X, Y).C = 0$ holds, then it is locally isometric to a Lorentzian unit sphere.

For a conformally symmetric Riemannian manifold [11], we have $\nabla C = 0$. Hence for such a manifold $R(X, Y).C = 0$ holds. Thus a conformally symmetric LP -Sasakian manifold M^n ($n > 3$) is locally isometric to a Lorentzian unit sphere [34].

For a conformally flat, quasi conformally flat and conformally symmetric LP -Sasakian manifold M^n , we have [34]

$$(2.2.3) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad X, Y, Z \in \Gamma(TM).$$

An arbitrary curve $\gamma : I \rightarrow M$, $\gamma = \gamma(s)$, in a LP -Sasakian manifold is called spacelike, timelike or null (lightlike), if all of its velocity vectors $\gamma'(s)$ are respectively spacelike, timelike or null (lightlike). If $\gamma(s)$ is a spacelike or timelike curve, we can reparametrize it such that $g(\gamma'(s), \gamma'(s)) = \varepsilon$ where $\varepsilon = 1$ if γ is spacelike and $\varepsilon = -1$ if γ is timelike, respectively. In this case $\gamma(s)$ is said to be unit speed or arclenght parametrization.

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\gamma(s)$ in a LP -Sasakian manifold. Then T, N, B_1, B_2 are respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. A spacelike or timelike curve $\gamma(s)$ is said to be parametrized by arclenght function s , if $g(\gamma'(s), \gamma'(s)) = \pm 1$.

Let $\gamma(s)$ be a curve in LP -Sasakian manifold parametrized by arclenght function s . Then for the curve γ the following Frenet equations are given in [35]:

Case I. γ is a spacelike curve:

Then T is a spacelike vector, so depending on the casual character of the principal normal vector N and the first binormal vector B_1 , we have the following Frenet formulas:

Case I.1. N and B_1 are spacelike;

$$(2.2.4) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying (2.2.4) the equations

$$g(T, T) = g(N, N) = g(B_1, B_1) = 1, \quad g(B_2, B_2) = -1.$$

Case I.2. N is spacelike, B_1 is timelike;

$$(2.2.5) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(T, T) = g(N, N) = g(B_2, B_2) = 1, \quad g(B_1, B_1) = -1.$$

Case I.3. N is spacelike, B_1 is null;

$$(2.2.6) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 satisfy the equations

$$\begin{aligned} g(T, T) &= g(N, N) = 1, & g(B_1, B_1) &= g(B_2, B_2) = 0, \\ g(T, N) &= g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = 0, & g(B_1, B_2) &= 1. \end{aligned}$$

Case I.4. N is timelike, B_1 is spacelike;

$$(2.2.7) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(T, T) = g(B_1, B_1) = g(B_2, B_2) = 1, \quad g(N, N) = -1.$$

Case I.5. N is null, B_1 is spacelike;

$$(2.2.8) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 satisfy the equations

$$g(T, T) = g(B_1, B_1) = 1, \quad g(N, N) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(B_1, B_2) = 0, g(N, B_2) = 1.$$

Case II. γ is a timelike curve:

In this case T is a timelike vector, so the Frenet formulae have the form

$$(2.2.9) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(N, N) = g(B_1, B_1) = g(B_2, B_2) = 1, \quad g(T, T) = -1.$$

3 Biharmonic curves in LP -Sasakian manifolds

In this section we characterize the spacelike and timelike proper biharmonic curves in a 4-dimensional conformally flat, quasi conformally flat and conformally symmetric Lorentzian para-Sasakian (LP -Sasakian) manifold.

Theorem 3.1 *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength. Suppose that $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ such that $g(T, T) = g(N, N) = g(B_1, B_1) = 1$ and $g(B_2, B_2) = -1$. Then $\gamma : I \rightarrow M$ is a proper biharmonic curve if and only if either γ is a circle with $k_1 = 1$, or γ is a helix with $k_1^2 + k_2^2 = 1$.*

Proof. Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold endowed with the structure (ϕ, ξ, η, g) and $\gamma : I \rightarrow M$ be a curve parametrized by arclength. Suppose that γ is a spacelike curve that is its velocity vector $T = \gamma'(s)$ is spacelike. Let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ , where N is the unit spacelike vector field in the direction $\nabla_T T$, B_1 is a unit spacelike and B_2 is a unit timelike vector. The tension field of γ is $\tau(\gamma) = \nabla_T T$. Then by using the Frenet formulas (2.2.4) and the equation (2.2.3) we obtain the Euler-Lagrange

equation of the bienergy:

$$\begin{aligned}
\tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\
&= \nabla_T^3 T - R(T, k_1 N)T \\
&= (-3k_1 k_1')T + (k_1'' - k_1^3 - k_1 k_2^2)N \\
&\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 - k_1 R(T, N)T \\
&= (-3k_1 k_1')T + (k_1'' - k_1^3 - k_1 k_2^2 + k_1)N \\
&\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 \\
&= 0.
\end{aligned}$$

where k_1 , k_2 and k_3 are respectively the first, the second and the third curvature of the curve $\gamma(s)$.

It follows that γ is a biharmonic curve if and only if

$$\begin{aligned}
k_1 k_1' &= 0, \\
k_1'' - k_1(k_1^2 + k_2^2 - 1) &= 0, \\
2k_1' k_2 + k_1 k_2' &= 0, \\
k_1 k_2 k_3 &= 0.
\end{aligned}$$

If we look for nongeodesic solutions, that is for biharmonic curves with $k_1 \neq 0$, we obtain

$$\begin{aligned}
k_1 &= \text{constant} \neq 0, \quad k_2 = \text{constant}, \\
k_1^2 + k_2^2 &= 1, \\
k_2 k_3 &= 0.
\end{aligned}$$

This completes the proof. ■

Theorem 3.2 *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength. Suppose that $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ such that $g(T, T) = g(N, N) = g(B_2, B_2) = 1$ and $g(B_1, B_1) = -1$. Then $\gamma : I \rightarrow M$ is a proper biharmonic curve if and only if either γ is a circle with $k_1 = 1$, or γ is a helix with $k_1^2 - k_2^2 = 1$.*

Proof. Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold endowed with the structure (ϕ, ξ, η, g) and $\gamma : I \rightarrow M$ be a curve parametrized by arclength. Suppose that γ is a spacelike curve that its velocity vector $T = \gamma'(s)$ is spacelike. Let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ , where N is the unit spacelike vector field in the direction $\nabla_T T$, B_2 is a unit spacelike and B_1 is a unit timelike vector. Since the tension field of γ is $\tau(\gamma) = \nabla_T T$ then by using the Frenet formulas given in (2.2.5) and the equation (2.2.3), we obtain

the biharmonic equation for γ :

$$\begin{aligned}
\tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\
&= \nabla_T^3 T - R(T, k_1 N)T \\
&= (-3k_1 k_1')T + (k_1'' - k_1^3 + k_1 k_2^2)N \\
&\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 - k_1 R(T, N)T \\
&= (-3k_1 k_1')T + (k_1'' - k_1^3 + k_1 k_2^2 + k_1)N \\
&\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 \\
&= 0.
\end{aligned}$$

where k_1 , k_2 and k_3 are respectively the first, the second and the third curvature of curve $\gamma(s)$.

It follows that γ is a biharmonic curve if and only if

$$\begin{aligned}
k_1 k_1' &= 0, \\
k_1'' - k_1(k_1^2 - k_2^2 - 1) &= 0, \\
2k_1' k_2 + k_1 k_2' &= 0, \\
k_1 k_2 k_3 &= 0.
\end{aligned}$$

If we look for nongeodesic solutions, that is for biharmonic curves with $k_1 \neq 0$, we obtain

$$\begin{aligned}
k_1 &= \text{constant} \neq 0, k_2 = \text{constant}, \\
k_1^2 - k_2^2 &= 1, \\
k_2 k_3 &= 0.
\end{aligned}$$

This completes the proof. ■

Theorem 3.3 *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength. Suppose that $\{T, N, B_1, B_2\}$ be a moving Frenet frame such that N is a spacelike and B_1 is a null vector. Then $\gamma : I \rightarrow M$ is a proper biharmonic curve if and only if $k_1 = 1$ and $\ln k_2(s) = -\int k_3(s) ds$.*

Proof. Let $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength on a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold M . Suppose that $\{T, N, B_1, B_2\}$ be a moving Frenet frame such that

$$g(T, T) = g(N, N) = 1, \quad g(B_1, B_1) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = 0, \quad g(B_1, B_2) = 1.$$

Then by using the Frenet equations given by (2.2.6), we have

$$\begin{aligned}
\tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\
&= \nabla_T^3 T - R(T, k_1 N)T \\
&= (-3k_1 k_1')T + (k_1'' - k_1^3 + k_1)N \\
&\quad + (2k_1' k_2 + k_1 k_2' + k_1 k_2 k_3)B_1
\end{aligned}$$

where k_1 , k_2 and k_3 are respectively the first, the second and the third curvature of curve $\gamma(s)$. From the biharmonic equation of γ above, we can say γ is a biharmonic curve if and only if

$$\begin{aligned} k_1 k_1' &= 0, \\ k_1'' - k_1^3 + k_1 &= 0, \\ 2k_1' k_2 + k_1 k_2' + k_1 k_2 k_3 &= 0. \end{aligned}$$

For biharmonic curves with $k_1 \neq 0$ that is if we investigate the nongeodesic solutions, we obtain

$$\begin{aligned} k_1 &= \mp 1, \\ k_2' + k_2 k_3 &= 0. \end{aligned}$$

Thus we have the assertion of the theorem. ■

Theorem 3.4 *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength. Suppose that $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ such that $g(T, T) = g(B_1, B_1) = g(B_2, B_2) = 1$ and $g(N, N) = -1$. Then $\gamma : I \rightarrow M$ is a biharmonic curve if and only if it is a geodesic of M .*

Proof. Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold endowed with the structure (ϕ, ξ, η, g) and $\gamma : I \rightarrow M$ be a curve parametrized by arclength. Suppose that γ is a spacelike curve that its velocity vector $T = \gamma'(s)$ is spacelike. Let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ , where N is the unit timelike vector field in the direction $\nabla_T T$, B_1 and B_2 are unit spacelike vectors. The tension field of γ is $\tau(\gamma) = \nabla_T T$. Then by using the tension field of γ , Frenet formulas in (2.2.7) and the equation (2.2.3) we obtain the Euler-Lagrange equation of the bienergy:

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\ &= \nabla_T^3 T - R(T, k_1 N)T \\ &= (3k_1 k_1')T + (k_1'' + k_1^3 + k_1 k_2^2)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 - k_1 R(T, N)T \\ &= (3k_1 k_1')T + (k_1'' + k_1^3 + k_1 k_2^2 + k_1)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 \\ &= 0. \end{aligned}$$

It follows that γ is a biharmonic curve if and only if

$$\begin{aligned} k_1 k_1' &= 0, \\ k_1'' + k_1(k_1^2 + k_2^2 + 1) &= 0, \\ 2k_1' k_2 + k_1 k_2' &= 0, \\ k_1 k_2 k_3 &= 0. \end{aligned}$$

If we look for nongeodesic solutions, that is for biharmonic curves with $k_1 \neq 0$, we obtain

$$\begin{aligned} k_1 &= \text{constant} \neq 0, k_2 = \text{constant}, \\ k_1^2 + k_2^2 &= -1, \\ k_2 k_3 &= 0. \end{aligned}$$

This shows that we have no nongeodesic solution for the biharmonic equation for the curve γ . ■

Theorem 3.5 *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength. Suppose that $\{T, N, B_1, B_2\}$ be a moving Frenet frame along γ such that N is a null vector. Then $\gamma : I \rightarrow M$ is a biharmonic curve if and only if γ is a geodesic of M .*

Proof. Let $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength on a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold M . Suppose that $\{T, N, B_1, B_2\}$ be a moving Frenet frame along the curve γ such that

$$g(T, T) = g(B_1, B_1) = 1, \quad g(N, N) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(B_1, B_2) = 0, \quad g(N, B_2) = 1.$$

If we consider the Frenet formulas given in (2.2.8), we obtain the biharmonic equation for the curve γ :

$$\begin{aligned} 0 = \tau_2(\gamma) &= (k_1'' + k_1 k_2 k_3 + k_1)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (-k_1 k_2^2)B_2 \end{aligned}$$

Then γ is a biharmonic curve if and only if

$$\begin{aligned} k_1'' + k_1 k_2 k_3 + k_1 &= 0, \\ 2k_1' k_2 + k_1 k_2' &= 0, \\ k_1 k_2^2 &= 0. \end{aligned}$$

Since γ is a spacelike curve with a null normal vector, k_1 can take only two values: 0 and 1. If we look for nongeodesic solutions, we get $k_2 = 0$. But from the first equation above, we have a contradiction such that $k_2 k_3 + 1 = 0$. So the only biharmonic spacelike curves on M with a null normal vector are the geodesics of M . ■

Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold. Since M is locally isometric to a Lorentzian unit sphere S_1^4 , by using the above theorems we shall give some characterizations for nongeodesic biharmonic curves in S_1^4 :

Proposition 3.6 *Let $\gamma : I \rightarrow S_1^4$ be a spacelike nongeodesic biharmonic curve parametrized by arclenght and $\{T, N, B_1, B_2\}$ be a Frenet frame along γ such that the principal normal vector N and first binormal vector B_1 are spacelike. Then*

$$(3.1) \quad \gamma^{(IV)} + 2\gamma'' + (1 - k_1^2)\gamma = 0.$$

Proof. From the Frenet formulas (2.2.4), by taking the covariant derivative of $\nabla_T N$ with respect to T , we have

$$\begin{aligned} \nabla_T^2 N &= -k_1 \nabla_T T + k_2 \nabla_T B_1 \\ &= -k_1^2 N + k_2(-k_2 N + k_3 B_2) \\ &= -(k_1^2 + k_2^2)N + k_2 k_3 B_2 \\ &= -N. \end{aligned}$$

If we use the Gauss equation of $S_1^4 \subset R_1^5$, that for any vector field X along γ is

$$\nabla_T X = X' + \langle T, X \rangle \gamma,$$

we get

$$\begin{aligned} \nabla_T^2 N &= \nabla_T[N' + \langle T, N \rangle \gamma] \\ &= \nabla_T N' \\ &= N'' + \langle T, N' \rangle \gamma \\ &= N'' + \langle T, \nabla_T N - \langle N, T \rangle \gamma \rangle \gamma \\ &= N'' + \langle T, \nabla_T N \rangle \gamma \\ &= N'' - k_1 \gamma \end{aligned}$$

and

$$N = \frac{1}{k_1}(\gamma'' + \gamma).$$

By substituting the above expressions of $\nabla_T^2 N$ and N in the equation $\nabla_T^2 N + N = 0$, we obtain the differential equation (3.1). ■

From Proposition 3.6, it is obvious that to find nongeodesic biharmonic curves in S_1^4 we must investigate the solutions of (3.1). By integrating the differential equation (3.1), we have

Theorem 3.7 *Let $\gamma : I \rightarrow S_1^4$ be a spacelike nongeodesic biharmonic curve parametrized by arclenght and $\{T, N, B_1, B_2\}$ be a Frenet frame along γ such that the principal normal vector N and first binormal vector B_1 are spacelike. Then we have two cases:*

- γ is a circle of radius $\frac{1}{\sqrt{2}}$;
- $\gamma(s) = (0, \frac{\cos(as)}{\sqrt{2}}, \frac{\sin(as)}{\sqrt{2}}, \frac{\cos(bs)}{\sqrt{2}}, \frac{\sin(bs)}{\sqrt{2}})$.

Proof. If $k_1 = 1$, then the general solution of (3.1) is

$$\gamma(s) = c_1 + c_2 s + c_3 \cos(\sqrt{2}s) + c_4 \sin(\sqrt{2}s).$$

Since $|\gamma|^2 = 1$ and $|\gamma'|^2 = 1$, we have $c_2 = 0$, while c_1, c_3, c_4 are constant vectors orthogonal to each other with $|c_1|^2 = |c_3|^2 = |c_4|^2 = \frac{1}{2}$. Then the solution becomes

$$\gamma(s) = (d_1, \frac{\cos(\sqrt{2}s)}{\sqrt{2}}, \frac{\sin(\sqrt{2}s)}{\sqrt{2}}, d_2, d_3),$$

with $-d_1^2 + d_2^2 + d_3^2 = \frac{1}{2}$. It is obvious that γ is a circle of radius $\frac{1}{\sqrt{2}}$.

If $0 < k_1 < 1$, then the general solution of (3.1) is

$$\gamma(s) = c_1 \cos(as) + c_2 \sin(as) + c_3 \cos(bs) + c_4 \sin(bs)$$

where $a = \sqrt{1 - k_1}$ and $b = \sqrt{1 + k_1}$. Since $|\gamma|^2 = 1$ and $|\gamma'|^2 = 1$, we obtain that the vectors c_i , $i = 1, 2, 3, 4$, are orthogonal to each other and $|c_1|^2 = |c_2|^2 = |c_3|^2 = |c_4|^2 = \frac{1}{2}$. Then the curve γ becomes

$$\gamma(s) = (0, \frac{\cos(as)}{\sqrt{2}}, \frac{\sin(as)}{\sqrt{2}}, \frac{\cos(bs)}{\sqrt{2}}, \frac{\sin(bs)}{\sqrt{2}}).$$

■

Proposition 3.8 *Let $\gamma : I \rightarrow S_1^4$ be a spacelike nongeodesic biharmonic curve parametrized by arclenght and $\{T, N, B_1, B_2\}$ be a Frenet frame along γ such that the principal normal vector N is spacelike and first binormal vector B_1 is timelike. Then*

$$(3.2) \quad \gamma^{(IV)} + 2\gamma'' + (1 - k_1^2)\gamma = 0.$$

If $k_1 = 1$, then it is obvious that the general solution of (3.2) is a circle of radius $\frac{1}{\sqrt{2}}$. If $k_1 > 1$, then the general solution of (3.2) is

$$\gamma(s) = c_1 e^{as} + c_2 e^{-as} + c_3 \cos(bs) + c_4 \sin(bs)$$

with $a = \sqrt{k_1 - 1}$ and $b = \sqrt{k_1 + 1}$. Here c_i , $i = 1, 2, 3, 4$, are constant vectors. Since $|\gamma|^2 = 1$ and $|\gamma'|^2 = 1$, by choosing

$$\begin{aligned} c_1 &= (1, 0, 0, 0, 1), \quad c_2 = (-1, \frac{\sqrt{7}}{4}, 0, 0, -\frac{3}{4}), \\ c_3 &= (0, 0, \frac{1}{2}, \frac{1}{2}, 0), \quad c_4 = (-\frac{\sqrt{7}}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, -\frac{\sqrt{7}}{\sqrt{2}}), \end{aligned}$$

such that

$$\begin{aligned} \langle c_1, c_1 \rangle &= \langle c_2, c_2 \rangle = 0, \\ \langle c_3, c_3 \rangle &= \langle c_4, c_4 \rangle = \frac{3}{b^2}, \\ \langle c_1, c_2 \rangle &= \frac{1}{a^2}, \\ \langle c_1, c_3 \rangle &= \langle c_1, c_4 \rangle = 0, \\ \langle c_2, c_3 \rangle &= \langle c_2, c_4 \rangle = 0, \\ \langle c_3, c_4 \rangle &= 0, \end{aligned}$$

with $a = 2$, $b = \sqrt{6}$, we obtain following special solution of differential equation (3.2)

$$\begin{aligned}\gamma(s) = & (e^{2s} - e^{-2s} - \frac{\sqrt{7}}{\sqrt{2}} \sin(\sqrt{6}s), \frac{\sqrt{7}}{4} e^{-2s} + \frac{1}{\sqrt{2}} \sin(\sqrt{6}s), \\ & \frac{1}{2} \cos(\sqrt{6}s), \frac{1}{2} \cos(\sqrt{6}s), e^{2s} - \frac{3}{4} e^{-2s} - \frac{\sqrt{7}}{\sqrt{2}} \sin(\sqrt{6}s)),\end{aligned}$$

which is a helix with $k_1 = 5$ and $k_2 = 2\sqrt{6}$.

Proposition 3.9 *Let $\gamma : I \rightarrow S_1^4$ be a spacelike nongeodesic biharmonic curve parametrized by arclenght and $\{T, N, B_1, B_2\}$ be a moving Frenet frame along γ such that the principal normal vector N is spacelike and first binormal vector B_1 is null. Then*

$$(3.3) \quad \gamma^{(IV)} + 2\gamma'' = 0.$$

It can be easily seen that the general solution of differential equation (3.3) is a circle of radius $\frac{1}{\sqrt{2}}$.

Now let us investigate the biharmonicity of a timelike curve in a 4-dimensional conformally flat, quasi conformally flat and conformally symmetric Lorentzian para-Sasakian (LP -Sasakian) manifold. We have,

Theorem 3.10 *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold and $\gamma : I \rightarrow M$ be a time-like curve parametrized by arclength. Then $\gamma : I \rightarrow M$ is a proper biharmonic curve if and only if either γ is a circle with $k_1 = 1$, or γ is a helix with $k_1^2 - k_2^2 = 1$.*

Proof. Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold endowed with the structure (ϕ, ξ, η, g) and $\gamma : I \rightarrow M$ be a curve parametrized by arclength. Suppose that γ is a time-like curve that is its velocity vector $T = \gamma'(s)$ is timelike. Let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ , where N is the unit spacelike vector field in the direction $\nabla_T T$, B_1 and B_2 are unit spacelike vectors. Then by using the Frenet formulas (2.2.9), we have:

$$\begin{aligned}\tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\ &= \nabla_T^3 T - R(T, k_1 N)T \\ &= (3k_1 k_1')T + (k_1'' + k_1^3 - k_1 k_2^2)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 - k_1 R(T, N)T \\ &= (3k_1 k_1')T + (k_1'' + k_1^3 - k_1 k_2^2 - k_1)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 \\ &= 0.\end{aligned}$$

It follows that γ is a biharmonic curve if and only if

$$\begin{aligned} k_1 k_1' &= 0, \\ k_1'' + k_1(k_1^2 - k_2^2 - 1) &= 0, \\ 2k_1' k_2 + k_1 k_2' &= 0, \\ k_1 k_2 k_3 &= 0. \end{aligned}$$

If we look for nongeodesic solutions, that is for biharmonic curves with $k_1 \neq 0$, we obtain

$$\begin{aligned} k_1 &= \text{const} \neq 0, k_2 = \text{const}, \\ k_1^2 - k_2^2 &= 1, \\ k_2 k_3 &= 0. \end{aligned}$$

■

Proposition 3.11 *Let $\gamma : I \rightarrow S_1^4$ be a timelike nongeodesic biharmonic curve parametrized by arclength. Then*

$$(3.4) \quad \gamma^{(IV)} - 2\gamma'' + (1 - k_1^2)\gamma = 0.$$

If $k_1 = 1$, then the general solution of (3.4) is

$$\gamma(s) = c_1 + c_2 s + c_3 e^{-\sqrt{2}s} + c_4 e^{\sqrt{2}s}$$

Here c_i , $i = 1, 2, 3, 4$, are constant vectors. Since $\langle \gamma(s), \gamma(s) \rangle = 1$ and $\langle \gamma'(s), \gamma'(s) \rangle = -1$, by choosing

$$\begin{aligned} c_1 &= \left(\frac{1}{\sqrt{2}}, 0, 0, 0, 1\right), \quad c_2 = (0, 0, 0, 0, 0), \\ c_3 &= \left(-1, \frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right), \quad c_4 = \left(1, -\frac{\sqrt{2}}{4}, \frac{1}{2\sqrt{2}}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right), \end{aligned}$$

such that

$$\begin{aligned} \langle c_1, c_1 \rangle &= \frac{1}{2} \\ \langle c_2, c_2 \rangle = \langle c_3, c_3 \rangle &= \langle c_4, c_4 \rangle = 0, \\ \langle c_1, c_2 \rangle = \langle c_1, c_3 \rangle &= \langle c_1, c_4 \rangle = 0, \\ \langle c_2, c_3 \rangle &= \langle c_2, c_4 \rangle = 0, \\ \langle c_3, c_4 \rangle &= \frac{1}{4}, \end{aligned}$$

we obtain following special solution of differential equation (3.4)

$$\begin{aligned} \gamma(s) &= \left(\frac{1}{\sqrt{2}} - e^{-\sqrt{2}s} + e^{\sqrt{2}s}, \frac{e^{-\sqrt{2}s}}{\sqrt{2}} - \frac{e^{\sqrt{2}s}}{2\sqrt{2}}, \right. \\ &\quad \left. \frac{e^{\sqrt{2}s}}{2\sqrt{2}}, \frac{e^{\sqrt{2}s}}{2}, 1 - \frac{e^{-\sqrt{2}s}}{\sqrt{2}} + \frac{e^{\sqrt{2}s}}{\sqrt{2}}\right), \end{aligned}$$

which is a circle.

If $k_1 > 1$, then the general solution of (3.4) is

$$\gamma(s) = c_1 e^{as} + c_2 e^{-as} + c_3 \cos(bs) + c_4 \sin(bs)$$

with $a = \sqrt{k_1 + 1}$ and $b = \sqrt{k_1 - 1}$. Here c_i , $i = 1, 2, 3, 4$, are constant vectors. Since again $\langle \gamma(s), \gamma(s) \rangle = 1$ and $\langle \gamma'(s), \gamma'(s) \rangle = -1$, by choosing

$$\begin{aligned} c_1 &= (1, 0, 0, 0, 1), \quad c_2 = (-1, \frac{\sqrt{7}}{4}, 0, 0, -\frac{3}{4}), \\ c_3 &= (0, 0, \frac{1}{2}, \frac{1}{2}, 0), \quad c_4 = (-\frac{\sqrt{7}}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, -\frac{\sqrt{7}}{\sqrt{2}}), \end{aligned}$$

such that c_1 and c_2 are null vectors and

$$\begin{aligned} \langle c_3, c_3 \rangle &= \langle c_4, c_4 \rangle = \frac{1}{b^2}, \\ \langle c_1, c_2 \rangle &= \frac{1}{a^2}, \\ \langle c_1, c_3 \rangle &= \langle c_1, c_4 \rangle = 0, \\ \langle c_2, c_3 \rangle &= \langle c_2, c_4 \rangle = 0, \\ \langle c_3, c_4 \rangle &= 0, \end{aligned}$$

with $a = 2$, $b = \sqrt{2}$, we obtain following special solution of differential equation (3.4)

$$\begin{aligned} \gamma(s) &= (e^{2s} - e^{-2s} - \frac{\sqrt{7}}{\sqrt{2}} \sin(\sqrt{2}s), \frac{\sqrt{7}}{4} e^{-2s} + \frac{1}{\sqrt{2}} \sin(\sqrt{2}s), \\ &\quad \frac{1}{2} \cos(\sqrt{2}s), \frac{1}{2} \cos(\sqrt{2}s), e^{2s} - \frac{3}{4} e^{-2s} - \frac{\sqrt{7}}{\sqrt{2}} \sin(\sqrt{2}s)), \end{aligned}$$

which is a helix with $k_1 = 3$ and $k_2 = 2\sqrt{2}$.

Remark 3.12 *In this paper we do not consider the null curves in a LP-Sasakian manifold. Because a null curve in a semi-Riemannian manifold can be considered as a 1-dimensional degenerate submanifold and some difficulties arise when the Laplacian operator is being defined in a degenerate submanifold. Hence the biharmonicity of a null curve thought of a 1-dimensional submanifold can not be defined by means of the variational problem.*

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